WYKŁAD 9 Z TEORII ERGODYCZNEJ (PO ANGIELSKU)

SIGNAL PROCESSES

Introduction

By a signal process we will understand a continuous time[1] process $X = (X_t)_{t\in\mathbb{R}}$ defined on a probability space (Ω, P) and assuming integer values, such that $X_0 = 0$ a.s., and with nondecreasing and right-continuous trajectories $t \mapsto X_t(\omega)$. We say that (for given $\omega \in \Omega$) a *signal* (or several simultneous signals) occurs at time t if the trajectory $X_t(\omega)$ jumps by a unit (or several units) at t.

A signal process is *homogeneous* if, for every finite set of times $t_1 < t_2 < \cdots < t_n$ and any $t_0 \in \mathbb{R}$, the joint distribution of the increments

$$X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2},\ \ldots,\ X_{t_n} - X_{t_{n-1}} \tag{1}$$

is the same as that of

$$X_{t_2+t_0} - X_{t_1+t_0},\ X_{t_3+t_0} - X_{t_2+t_0},\ \ldots,\ X_{t_n+t_0} - X_{t_{n-1}+t_0}.$$

The most basic example of a stationary signal process is the Poisson process. It is characterized by two properties: 1. the increments as described in (1) are independent, and 2. jumps by more the one unit have probability zero. These properties imply that the distribution of X_1 is the Poisson distribution with some parameter $\lambda \geq 0$ (i.e., $P\{X_1 = k\} = e^{-\lambda}\frac{\lambda^k}{k!}$, $k = 0, 1, \ldots$). The parameter λ is called the *intensity* and equals the expected number of signals per unit of time.

Given a signal process (X_t), by the *waiting time* we will understand the random variable defined on Ω as the time of the first signal after time 0:

$$V(\omega) = \min\{t : X_t(\omega) \geq 1\}.$$

We denote by F (or F_X if the reference to the process is needed) the distribution function of the waiting time $\bar{V}$.

Special flows versus signal processes

We will first recall some basic information about *flows* - i.e., dynamical systems with continuous time. By a flow $(\Omega, \Sigma, \mu, T_t)$ we will understand the action of a group of measurable and measure μ-preserving transformations T_t $(t \in \mathbb{R})$ on a probability space (Ω, Σ, μ), satisfying the composition rule $T_{t+s} = T_t \circ T_s$, and such that the trajectories $t \mapsto T_t(\omega)$ are measurable for almost every $\omega \in \Omega$.

Now suppose we have a probability space (B, ν) and a measure ν-preserving bijection $\phi : B \to B$. There is a specific method of building a flow based on a the single map ϕ and a nonnegative integrable function f defined on B. Let $\theta = \int f d\nu$.

[1] also discrete time, when the increment of time is very small

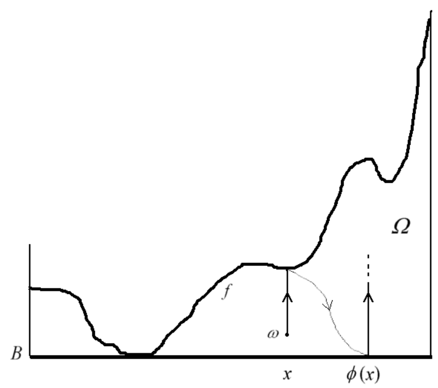
We will describe so-called *special flow* for which f has the name of a *roof function*. The space Ω for the flow is defined as the area below the graph of f:

$$\Omega = \{(x, y) \in B \times \mathbb{R} : 0 \le y < f(x)\},$$

the measure is $\mu = \nu \times \frac{\eta}{\theta}$, where η is the Lebesgue measure on $\mathbb{R}$. Clearly, μ is a probability measure on Ω. The special flow is now defined as follows:

$$T_t(x, y) = \begin{cases} (x, y + t) & y + t < f(x) \\ (\phi(x), 0) & y + t = f(x). \end{cases}$$

For t larger than $f(x) - y$ we divide t into smaller pieces and apply the above formula and the composition rule.



Each point $\omega = (x, y)$ travels upward with unit speed until it reaches the "roof". Then it jumps to the "floor" at $(\phi(x), 0)$, continues upward, and so on.

Such a special flow gives raise to a signal process X defined on the same space (Ω, μ): for each ω the signals are identified with the visits of the trajectory of ω at the floor. In other words

$$X_t(\omega) = \#\{s \in (0, t] : T_s(\omega) \in B \times \{0\}\}.$$

We now compute the intensity λ of this process:

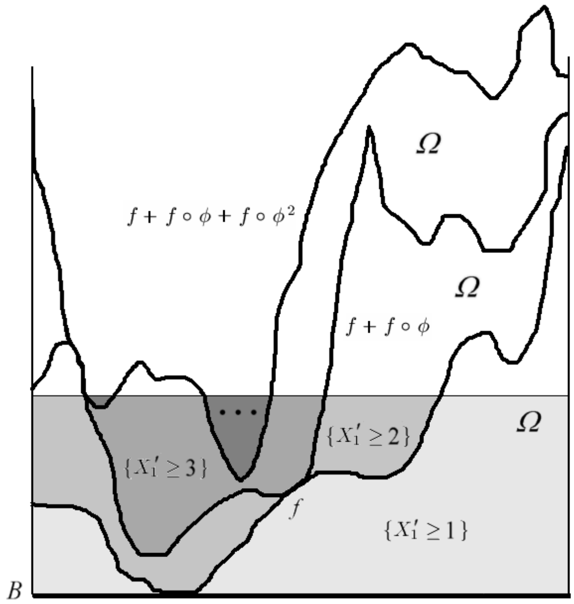
Theorem.

$$\lambda = \theta^{-1}.$$

Proof. First we replace X_1 by $X_1' = X_0 - X_{-1}$ (it counts the signals between times -1 and 0). Then

$$\lambda = \mathrm{E}(X_1) = \mathrm{E}(X_1') = \sum_{k=1}^{\infty} k\mu(\{X_1' = k\}) = \sum_{k=1}^{\infty} \mu(\{X_1' \ge k\}).$$

Now we draw the diagram with the graphs of the functions $g_0 = 0$, $g_1 = f$, $g_2 = f + f \circ \phi$, $g_3 = f + f \circ \phi + f \circ \phi^2$, etc.

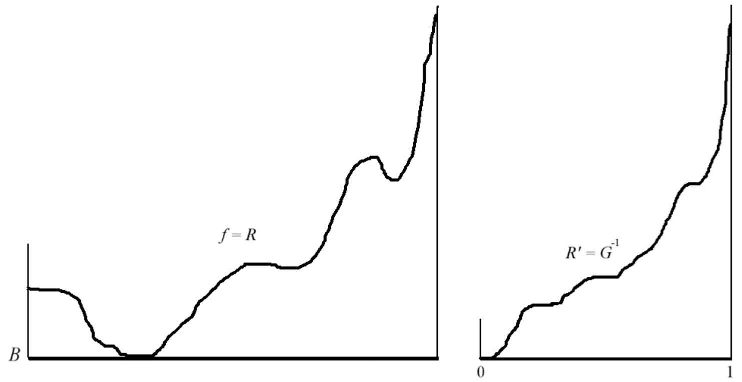


This diagram shows multiple copies of the set Ω; one between each pair of the graphs. On this diagram each point (x, y) travels up with unit speed, and, meeting a graph, instead of jumping to the floor, it passes to another copy of Ω. The signals correspond to such passages. Thus, in order to count the signals between times 0 and -1 for such point, we need to see how many graphs are there between (x, y) and $(x, y - 1)$. It is now easy to see that the sets $\{X_1' \geq k\}$ correspond (each in a different copy of Ω) to the areas between the graphs of g_{k-1} and g_k and below the line $y = 1$ (see the figure). Together they add up to the full rectangle between the lines 0 and 1, whose measure μ is θ^{-1} (recall that on the second coordinate we use the Lebesgue measure times θ^{-1}). Eventually we have obtained that

$$\lambda = \theta^{-1}.$$

□

It is important, that in fact every signal process with intensity λ can be modeled in this way, using a special flow, under a roof function of integral λ^{-1}. In this approach we have a new random variable R defined on the floor (B, μ) and called the *return time*. This is simply the roof function f and its expected value is λ^{-1}. If G denotes the distribution function of R, then the inverse R' of G (with intervals of constancy turned into jumps and vice-versa), treated as a variable on the unit interval with the Lebesgue measure, has the same distribution as R.

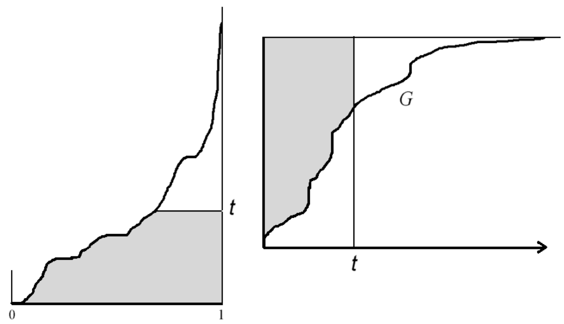


We can replace R by the inverse of G defined on the interval [0,1].

We can now derive a relation between the distribution function G of R and F of the waiting time V. Fix some $t > 0$. We have

$$F(t) = \mu\{V \leq t\} = \mu\{X_t \geq 1\} = \mu\{X'_t \geq 1\}$$

(where, like before, $X'_s = X_0 - X_{-s}$). On the picture of Ω (the area below the graph of f), the latter set is the region below the (horizontal) line $y = t$ which can be as well drawn on the graph of R'. On the graph of the distribution function G, this corresponds to the area above G and to the left of the (vertical) line $y = t$.



Thus, taking into account the factor $\theta^{-1} = \lambda$ in evaluating probabilities from areas on the diagram, we obtain

$$F(t) = \lambda \int_0^t 1 - G(y)\, dy.$$

As an immediate corollary, we get that F is always a continuous and concave function.